Gauss-Jordan Elimination

To solve a matrix using Gauss-Jordan elimination, go column by column.

First, get a 1 in the first row of the first column.

$$\left|\begin{array}{ccc|c} 1 & & & \\ & & & \\ & & & \end{array}\right|$$

Then, get zeros as the remaining entries of that column. The order in which you get the remaining zeros does not matter.

$$\left|\begin{array}{ccc|c} 1 & & & \\ 0 & & & \\ 0 & & & \end{array}\right|$$

Once this is done, your first column is complete. To begin your second column, get a 1 in the second row.

$$\left|\begin{array}{ccc|c} 1 & & & \\ 0 & 1 & & \\ 0 & & & \end{array}\right|$$

Then, get zeroes as the remaining entries of that column.

$$\left|\begin{array}{ccc|c} 1 & 0 & & \\ 0 & 1 & & \\ 0 & 0 & & \end{array}\right|$$

For the last column, we begin by getting a 1 in the last row.

$$\left|\begin{array}{ccc|c} 1 & 0 & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \end{array}\right|$$

We then proceed by getting zeroes as the remaining entries.

$$\left|\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right|$$

Getting 1's:

- Row swap (*you can only do a row swap on your first step)
  [option 1 in row ops]
- Multiplying row by reciprocal.
  [option 3 in row ops]

Getting 0's:

- Multiply by additive inverse and/or add rows.
  [option 2 or 4 in row ops]

Three Possible Outcomes (Examples):

1) One Solution:

$$\left|\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right|$$

$x = 5$, $y = 2$, and $z = 4$. The solution is (5, 2, 4).

2) Infinite Solutions (dependent system):

$$\left|\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right|$$

The equations are:
$x + 2z = 3$ and $y - 3z = 4$

Solve for x and y:
$x = 3 - 2z$ and $y = 4 + 3z$

Thus, the solution is:
$(3 - 2z, 4 + 3z, z)$ with z = any real number

3) No Solution (inconsistent system):

$$\left|\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right|$$

Here, this matrix tells us that 0 = 4 which we know is false. Thus, there is no solution.

Example:

Solve the system represented by the matrix: $\left|\begin{array}{ccc|c} 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{array}\right|$

The first step is to get a 1 in the first row of the first column.

Since there is a 1 in the second row, we'll do a row swap.
[Rowops: option 1]

$$R_1 \xleftrightarrow{SWAP} R_2$$

$$\left|\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \\ 2 & 3 & 1 & 4 \end{array}\right|$$

The second step is to get zeros in the remaining cells of the first column.

We will do this by using additive inverses.
[Rowops: option 4]

$$-3R_1 + R_2 \xrightarrow{YIELDS\ THE\ NEW} R_2$$

$$\left|\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -7 & -11 \\ 2 & 3 & 1 & 4 \end{array}\right|$$

[Rowops: option 4]

$$-2R_1 + R_3 \xrightarrow{YIELDS\ THE\ NEW} R_3$$

$$\left|\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -7 & -11 \\ 0 & -1 & -5 & -4 \end{array}\right|$$

Now that we are done with the first column, we move to the second column. The next step is to get a 1 in the second row of the second column.
[Rowops: option 3]

$$-\frac{1}{2}R_2 \xleftrightarrow{YIELDS\ THE\ NEW} R_2$$

$$\left|\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 7/2 & 11/2 \\ 0 & -1 & -5 & -4 \end{array}\right|$$

We now get zeros in the remaining entries of column 2.

[Rowops: option 2]

$$R_2 + R_3 \xrightarrow{YIELDS\ THE\ NEW} R_3$$

$$\left|\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 7/2 & 11/2 \\ 0 & 0 & -3/2 & 3/2 \end{array}\right|$$

[Rowops: option 4]

$$-2R_2 + R_1 \xrightarrow{YIELDS\ THE\ NEW} R_1$$

$$\left|\begin{array}{ccc|c} 1 & 0 & -4 & -7 \\ 0 & 1 & 7/2 & 11/2 \\ 0 & 0 & -3/2 & 3/2 \end{array}\right|$$

Now that we are done with the second column, we move to the third column. We begin by getting a 1 in the third row. **[Rowops: option 3]**

$$-\frac{2}{3}R_3 \xleftrightarrow{YIELDS\ THE\ NEW} R_3$$

$$\left|\begin{array}{ccc|c} 1 & 0 & -4 & -7 \\ 0 & 1 & 7/2 & 11/2 \\ 0 & 0 & 1 & -1 \end{array}\right|$$

Finally, we get zeros in the remaining entries of column 3.

[Rowops: option 4]

$$-\frac{7}{2}R_3 + R_2 \xrightarrow{YIELDS\ THE\ NEW} R_2$$

$$\left|\begin{array}{ccc|c} 1 & 0 & -4 & -7 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -1 \end{array}\right|$$

[Rowops: option 4]

$$4R_3 + R_1 \xrightarrow{YIELDS\ THE\ NEW} R_1$$

$$\left|\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -1 \end{array}\right|$$